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On the Investment Game

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Abstract

We examine the *investment game* analyzed by Fudenberg and Levine (1994) where long-run players (capitalists) live infinitely, borrow investment goods from short-lived investors, and produce an output. They argued that the maximal sum of equilibrium payoffs can attain a first best payoff when discount factor is close to one and the number of capitalists is large enough. However, we show that the supremum of the sum of equilibrium payoffs is different from the first best payoff in general. Furthermore, if the probability of success of production is sufficiently high, the supremum is unchangeable with respect to the number of capitalists.

key words: Repeated game, Imperfect information, Equilibrium characterization.

JEL classifications : C72, D92

1 Introduction

We examine the *investment game* analyzed by Fudenberg and Levine (1994). In the game, long-run players are called capitalists who live infinitely, borrow investment goods from short-lived investors, produce an output from the investment goods, and pay a dividend to short-run players. This production process is uncertain to produce high output. Short-run players are called investors who play at only one stage, invest with a capitalist at first round of the stage, and receive a dividend from him at second round of

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the stage. Each investor imperfectly observes the actions of capitalists. If a capitalist pay no dividend, all investors cannot distinguish whether he schedules to pay no dividend or the realized output is zero while he wishes to pay a positive dividend.

Fudenberg and Levine proposed an algorithm *maximum score method* to characterize the set of equilibrium payoffs with sufficiently little discounting in the class of infinitely repeated games with several short-run players. By using this method, they derived the limit equilibrium payoffs of the investment game when the number of capitalist is one. In their Corollary 8.1 Fudenberg and Levine (1994, p.130) argued that if there exist sufficiently many capitalists, then the maximal sum of equilibrium payoffs for capitalists can attain a first best payoff. However, we show that the supremum of the sum of equilibrium payoffs is in general different from the first best payoff by using their equilibrium characterization theorem. We also show that when the probability of high output is sufficiently high, the supremum is unchangeable regardless of the number of capitalists. In the middle range the probability, the more the number of capitalists increases, the more the supremum increases up to an efficient point. When the probability of success is low, no short-run player has incentive to invest her goods with any capitalist any more.

Given a repeated game such as the investment game, let V be feasible and individually rational payoffs for long-run players. Standard Folk theorem asserts that all points of V can be equilibrium outcome when the dimension of V is equal to the number of long-run players and their common discount factor is close to one.(1) Since the game has short-run players, we must consider their incentives. Let V^* be a set of payoff vectors with incentive constraints for short-run players in addition to feasibility and individual rationality. Since V^* is a proper subset of V except under trivial situations, Folk theorem does not hold in this kind of games. Can all points of V^* arise as equilibrium outcomes? By the imperfect information such as capitalist's dividend payment schedule in the investment game, it is not true in general. In order to characterize the equilibrium outcome, the maximum score method derives a set Q which is a subset of V^*. Fudenberg and Levine (1994) proved that all interior points of Q can be equilibrium point when the dimension of Q is equal to the number of long-run players and they are sufficiently patient. The sufficient condition for their equilibrium theorem corresponds to the ordinary sufficient condition for standard Folk theorem in repeated games without short-run players and imperfect information. Since this enable us to measure to what extent these two factors contract the equilibrium set, their equilibrium theorem is a seminal result in area of repeated games.

This paper is organized as follows. In Section 2 the stage game of the investment game is formulated. In Section 3 we define a repeated game and introduce the maximum score method in Fudenberg and Levine (1994). In Section 4 we provide the limit equilibrium payoffs in the investment game. Section 5 summarizes our results.

2 Stage Game

There exist two classes of players. One is long-run player who is called capitalist, lives infinitely, and has a production technology with uncertainty. We assume that all capitalists are identical. Let $LR := \{1, 2, \ldots, n\}$ be the long-run players set. Another is short-run player who is called investor and lives one stage only. Stage game consists of these two rounds. She is born with one unit initial endowment at first round. She decides to invest her good with a capitalist or to consume it at first round. She is endowed no good at second round. If the capitalist succeeds in high output and pays dividend to her according to his dividend payment schedule, she consumes her dividend at second round. When the short-run players exit at the end of second round of a stage, new short-run players enter the game at the beginning of first round of the next stage. Short-run player is the same type but different player. In stage game, by abuse of terminology, we call them short-run player. Let $SR := \{n+1, n+2, \ldots, \underline{x}\nu\}$ be the short-run players type space. We assume that all short-run players are identical investor. We will explain the symbols $\underline{x}$ and ν later. We define a production technology which all capitalists have. If a capitalist gathers investment goods x greater than or equal to a threshold level $\underline{x}$, then output levels are xf/p with probability p (high output case) or 0 with probability $1-p$ (low output case). Otherwise, output is 0. Expected output f is greater than 1. All capitalists' successes are independent. The threshold $\underline{x}$ is greater than 1 so that even if the investors have the technology, they cannot produce hight output by their own production. Since there are $\underline{x}\nu$ investors, at most ν different capitalists can produce high output.

At the first round a young investor $j \in SR$ must decide to invest her endowment with a capitalist or to consume it immediately. Since $|LR| = n$, she has $n+1$ available actions. At second round the old investor only consumes a dividend if any. Let A_j for $j \in SR$ be her action space. It consists of $(a_j^j, a_j^1, \ldots, a_j^n)$ as follows. If j consumes her endowment, then $a_j^j = 1$. Otherwise, $a_j^j = 0$. If j invests with $i \in LR$, then $a_j^i = 1$. Otherwise, $a_j^i = 0$. Let $\mathcal{A}_j$ be her mixed action space. We denote by A_{SR} and $\mathcal{A}_{SR} := \times_{j \in SR} \mathcal{A}_j$ the pure and mixed action profile space for short-run players, respectively. We use a_{SR}

and α_{SR} as their generic elements. Given a short-run player mixed action α_j we denote the probability of investing with long-run player i by $\alpha_j(a_j^i)$. We assume that all players can perfectly observe the short-run players' actions.

At each first round each capitalist i decides what proportion of realized output a_i is set aside to pay back as dividend payment. If he chooses a_i and the realized output is z, then $a_i z$ is equally divided among the investors who invest with him. We assume that the action a_i is his private information. He consumes the remainder $(1 - a_i)z$. Since his plan of dividend payment is his private information, so is his own consumption. We denote the entire action space for long-run player by $A_i = \{a_i^1 = 0, a_i^2, \ldots, a_i^m\}$ such that each element is less than one. Each capitalist can take greedy action $a_i^1 = 0$ to pay no dividend payment to investors. We denote by A_{LR} and $\mathcal{A}_{LR}$ the sets of long-run players' pure action profile and mixed action profile, respectively. We use a_{LR} and α_{LR} as their generic elements. We denote by A and $\mathcal{A}$ the set of all players' pure and mixed action profile, respectively. After each investor makes an investment decision and each capitalist chooses a dividend payment schedule at the first round, each production occurs according to the production technology. If an output is high, a capitalist returns a dividend to his investors and consumes the rest in accordance with his payment plan at second round. If an output is low, he refunds no dividend to his investors and consumes nothing. Let $J_i(a_{SR})$ be the amount of input to $i \in LR$ when the short-run players use the pure action profile a_{SR}. By the definition of the short-run players' action and the one unit endowment, $J_i(a_{SR}) = \sum_{j \in SR} a_j^i$. Let $P_i(a_{SR})$ be the probability of enough investment for capitalist $i \in LR$ to be able to make high output conditional on a_{SR}. Since every short-run player's action is perfectly observable, the value of $P_i(a_{SR})$ is 0 or 1, i.e., if $\sum_{j \in SR} a_j^i \geq \underline{x}$, then $P_i(a_{SR}) = 1$. Otherwise, $P_i(a_{SR}) = 0$.

We explain the information structure of the investment game. The level of output and the amount consumed by capitalist i are his own private information. But his realized dividend payment is observable. The other players can statistically infer the capitalists' actions by the realized dividend payments. Let y_i be his realized payment per investors who invest with i at first round. We call y_i the *public information* corresponding to $i \in LR$. Let Y_i be his entire public information. Since each capitalist equally divides the part of his output among his investors according to his plan of payment a_i, we can naturally assume that $Y_i = A_i$ and $y_i^k = a_i^k$ for $k = 1, \ldots, m$. Given a pure action profile (a_i, a_{SR}), let $\pi_i(y_i | a_i, a_{SR})$ be the probability of the i's public information y_i.

If he uses greedy action $a_i^1 = 0$, then $\pi_i(y_i^1|a_i^1, a_{SR}) = 1$. If he uses $a_i \neq a_i^1$, then $\pi_i(y_i|a_i, a_{SR}) = P_i(\alpha_{SR})p$ and $\pi_i(y_i^1|a_i, a_{SR}) = 1 - P_i(\alpha_{SR}) + (1-p)P_i(\alpha_{SR})$. In order to make high output the capitalist must gather enough investments and Nature must choose success for him. In this case the realized payment per his investors is $y_i = a_i$. Low output cases are divided into two cases. The capitalist can not induce investors to invest with him. Although he can collect enough investments, the production fails with probability $1-p$. In this case the realized payment is $y_i^1 = 0$. The players other than i cannot distinguish whether the output is low or he uses the greedy action $a_i^1 = 0$. It can be said that each long-run player is subject to moral hazard since he can choose the greedy action. We also denote the probability of y_i conditional on (a_i, α_{SR}) by $\pi_i(y_i|a_i, \alpha_{SR})$. Let $y := (y_1, \ldots, y_n)$ be a public information profile. Given a pure action profile $a \in A$, let $\pi(y|a)$ be the probability of the public information profile y. It is easily seen that $\pi(y|a) = \pi_1(y_1|a_1, a_{SR})\pi_2(y_2|a_2, a_{SR}) \cdots \pi_n(y_n|a_n, a_{SR})$. This property of information structure is called *product structure.*

Since a capitalist i's payoff is not influenced by the other capitalists' dividend payments, his stage game payoff g_i only depends on the observable actions for him.

$$
\begin{aligned}
g_i(a_i, a_{SR}) &:= (1-a_i)P_i(a_{SR})J_i(a_{SR})f \\
&= \begin{cases} (1-a_i)\sum_{j\in SR} a_j^i f & \text{if } \sum_{j\in SR} a_j^i \geq \underline{x} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

If y is realized, the pure action payoff for investor $j \in SR$ is

$$
r_j(a_j, y) := \sum_{i\in LR} a_j^i y_i f/p + a_j^j.
$$

By using the probabilities $\pi(y|a)$, the expected payoff for investor j is

$$
\begin{aligned}
g_j(a) &:= \sum_{y\in Y} \pi(y|a) r_j(a_j, y) \\
&= \sum_{y\in Y}\sum_{i\in LR} \pi_i(y_i|a_i, a_{SR}) a_j^i y_i f/p + a_j^j \quad \text{by product structure} \\
&= \sum_{i\in LR} a_j^i P_i(a_{SR}) a_i f + a_j^j \quad \text{by definition of } \pi_i.
\end{aligned}
$$

Any investor can consume directly at first round ($a_j^j = 1$), irrespective of the long-run players' actions. Clearly the minimax value for investors is 1. If a capitalist wishes investors to invest with him, he must assure them at least this minimax payoff level. To attain his purpose he must choose a_i such that $a_i f \geq 1$. We denote by $\underline{a} := \min_{a'}\{a' \in A_i \mid a'f \geq 1\}$ the *minimum action compatible with investor's incentive*. If capitalists' available actions and the expected output f were sufficiently large, then there might exist the minimum action in A_i. We assume the existence of the minimum action $\underline{a}$. We denote by B the investors' best response correspondence from capitalists' mixed actions to investors' mixed actions. For given mixed action profile α^* belonging to the graph of B, every investor j maximizes her payoff by using α_j^*. We also denote by $\hat{B}$ the short-run players' best response correspondence from long-run players' *pure* actions to short-run players' mixed actions.

Let $\boldsymbol{g}(a) = (g_1(a), \ldots, g_n(a))^{\mathrm{T}}$ be the payoff vector for long-run players under pure action profile $a \in A$. We denote the minimax value for $i \in LR$ by $\underline{v}_i := \min_{\alpha_{-i}} \max_{a_i} g_i(a_i, \alpha_{-i})$. Let $\underline{\boldsymbol{v}} := (\underline{v}_1, \ldots, \underline{v}_n)^{\mathrm{T}}$ be the minimax value vector. In general, the feasible and individually rational payoff set is

$$V := \{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{v} \in \mathrm{co}\, \boldsymbol{g}(A) \text{ and } \boldsymbol{v} \geq \underline{\boldsymbol{v}}\}.$$

Since our game has short-run players, we must consider their incentives. In general, the set of feasible, individually rational, and incentive compatible payoffs is

$$V^* := \{\boldsymbol{v} \in \mathbb{R}^n \mid \boldsymbol{v} \in \mathrm{co}\, \boldsymbol{g}(\mathrm{graph}\, \hat{B}) \text{ and } \boldsymbol{v} \geq \underline{\boldsymbol{v}}\}.$$

Lemma 2.1 In the investment game,

$$V = \left\{ \boldsymbol{v} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i \in LR} v_i \leq \underline{x}\nu f \right\} \text{ and } V^* = \left\{ \boldsymbol{v} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i \in LR} v_i \leq \underline{x}\nu f(1-\underline{a}) \right\}. \quad \square$$

Proof: It is easily seen that the minimax value is 0 for all $i \in LR$ since all short-run players can consume their endowments at first round. The maximum payoff for $i \in LR$ is $\underline{x}\nu f$ since he chooses the greedy action ($a_i^1 = 0$) and all short-run player invest with him, i.e., $a_j^i = 1$ for all j. In this case the other long-run players' payoffs are 0. Therefore, $\underline{x}\nu f \boldsymbol{e}_i$ belongs to V for all $i \in LR$ where $\boldsymbol{e}_i$ is the standard unit vector in which only ith coordinate is one in $\mathbb{R}^n$. We claim that the extremal points of V are $\boldsymbol{0}$ and $\underline{x}\nu f \boldsymbol{e}_i$ for all $i \in LR$. If there exists a payoff vector $\boldsymbol{v}$ corresponding to a pure action such that $\sum_{i \in LR} v_i > \underline{x}\nu f$, this implies that the sum of the investment goods is at least more than $\underline{x}\nu$, a contradiction. By the condition on the pure action $a_i \in [0,1]$, any payoff vector $\boldsymbol{v}$

must fulfill $\sum_{i \in LR} v_i \leq \underline{x}\nu f$ and $v_i \geq 0$ for all $i \in LR$. The claim holds. It is easily seen that the convex hull of these vectors is

$$V = \left\{ \boldsymbol{v} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i \in LR} v_i \leq \underline{x}\nu f \right\}.$$

If we maximize capitalist i's payoff over all players' actions satisfying investors' incentive conditions, his payoff is $\underline{x}\nu f(1-\underline{a})$ and the others are 0. Therefore, each vector $\underline{x}\nu f(1-\underline{a})\boldsymbol{e}_i$ fulfills the short-run's incentive condition. By the similar way above,

$$V^* = \left\{ \boldsymbol{v} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i \in LR} v_i \leq \underline{x}\nu f(1-\underline{a}) \right\}. \qquad \blacksquare$$

The set of payoff vectors V^* for two capitalists case is depicted in Figure 1. Note that $\dim V = \dim V^* = n$. The investment game satisfies the full-dimensionality condition. Clearly, the unique Nash equilibrium payoff vector for capitalists in the stage game is $\mathbf{0}$. All investors do not invest and all capitalists choose an action a_i such that $a_i f < 1$. Especially, in one of Nash equilibria no capitalist repays ($a_i^1 = 0$). Note that the stage game Nash equilibrium payoff vector $\mathbf{0}$ is equal to the minimax value.

3 Repeated Game and Equilibrium Characterization

The stage game is infinitely repeated $t = 1, 2, \ldots$ by the same capitalists and the same investor types. By using the common discount factor $\delta \in [0, 1)$ each capitalist maximizes the average of the sum of discounted stage game payoffs. At any time t all capitalists and current investors can observe the dividend payments up to time $t-1$ and all past investors' actions. This commonly observable information is called *public history*. We restrict our strategy to *public strategy* by which any action at any period depends only on this period's public history. Our equilibrium concept is *perfect public equilibrium* (PPE). A PPE is a profile of public strategies such that at any period and for any this period's history the strategies are a Nash equilibrium from that period on.

Let $E(\delta)$ be the set of PPE payoff vectors for capitalists under a discount factor δ. In order to characterize $\lim_{\delta \to 1} E(\delta)$ for general repeated games, Fudenberg and Levine (1994) proposed *maximum score method*. In this method, we first solve the following linear programming problem for an action profile α and a direction $\boldsymbol{\lambda} \in \mathbb{R}^n$.

Definition 3.1 (linear programming problem) Given a mixed action profile $\alpha \in \mathcal{A}$, a directional vector $\boldsymbol{\lambda} \in \mathbb{R}^n$, and a discount factor $\delta \in [0, 1)$,

$$k^*(\alpha, \boldsymbol{\lambda}) := \max_{\boldsymbol{v}, \boldsymbol{w}(y)} \langle \boldsymbol{\lambda}, \boldsymbol{v} \rangle \quad \text{subject to} \tag{1}$$

$$v_i = (1-\delta) g_i(a_i,\ \alpha_{-i}) + \delta \sum_{y \in Y} \pi(y | a_i,\ \alpha_{-i}) w_i(y) \tag{1a}$$

$$\text{for } a_i \in \operatorname{support} \alpha_i \text{ for } i \in LR,$$

$$v_i \geq (1-\delta) g_i(a_i,\ \alpha_{-i}) + \delta \sum_{y \in Y} \pi(y | a_i,\ \alpha_{-i}) w_i(y) \tag{1b}$$

$$\text{for } a_i \notin \operatorname{support} \alpha_i \text{ for } i \in LR,$$

$$\langle \boldsymbol{\lambda}, \boldsymbol{v} \rangle \geq \langle \boldsymbol{\lambda}, \boldsymbol{w}(y) \rangle \quad \text{for all } y \in Y. \ \square \tag{1c}$$

Note that $\boldsymbol{w} : Y \to \mathbb{R}^n$, and $\boldsymbol{w}(y) = (w_1(y), \ldots, w_n(y))^{\mathrm{T}} \in \mathbb{R}^n$ for each $y \in Y$. This problem derives an optimal value $k^*(\alpha, \boldsymbol{\lambda})$. Second, we maximize $k^*(\alpha, \boldsymbol{\lambda})$ over $\alpha \in \operatorname{graph} B$. The maximized value $k^*(\boldsymbol{\lambda})$ is called *maximal score in direction* $\boldsymbol{\lambda}$. Third, we make a closed half-space of $\boldsymbol{\lambda}$ and $k^*(\boldsymbol{\lambda})$. Let us define a *maximal half-space in direction* $\boldsymbol{\lambda}$ by $H^*(\boldsymbol{\lambda}) := \{\boldsymbol{v} \in \mathbb{R}^n \mid \langle \boldsymbol{\lambda}, \boldsymbol{v} \rangle \leq k^*(\boldsymbol{\lambda})\}$. Finally, we compute Q $(\subset \mathbb{R}^n)$ such that $Q := \bigcap_{\boldsymbol{\lambda} \in \mathbb{R}^n} H^*(\boldsymbol{\lambda})$. Fudenberg and Levine (1994) proved the following theorem.

Theorem 3.1 (Fudenberg and Levine (1994)) For any game, if $\dim Q = n$, then $\lim_{\delta \to 1} E(\delta) \supset \operatorname{int} Q$. $\square$

To derive equilibrium vectors this algorithm seems to be complicated. However, the investment game has many tractable features. To state those we must distinguish between coordinate and regular directions. The direction $\boldsymbol{\lambda}$ is *coordinate* if exactly one component is nonzero and it is *regular* if at least two components are nonzero. The coordinate direction is *negative* if it has negative component and it is *positive* if it has positive component.

Lemma 3.1 (Fudenberg and Levine (1994)) In the investment game, if $\boldsymbol{\lambda}$ is either regular or negative coordinate direction, then $H^*(\boldsymbol{\lambda}) \supset V^*$. Furthermore, the maximal score $k^*(-\boldsymbol{e}_i)$ in the latter case is 0 for all $i \in LR$. $\square$

Since $E(\delta)$ is a subset of V^*, any maximal half-space in these directions put no constraint on Q. Then, we may only compute maximal scores in positive coordinate directions.

4 Equilibrium for the Investment Game

Thanks to Lemma 3.1,we only calculate each maximal score $k^*(\boldsymbol{e}_1)$ that maximizes player 1's payoff. Note that $\langle \boldsymbol{e}_1, \boldsymbol{v}\rangle = v_1$ and $\langle \boldsymbol{e}_1, \boldsymbol{w}(y)\rangle = w_1(y)$. Let us consider the shortcut to derivation of Q. In order to maximize v_1 in the problem (1), it is necessary that all investors must invest with him. By Lemma 2.1, any pure action $a_1 \geq \underline{a}$ is sufficient for any short-run player to invest with player 1. In this case all the short-run players may invest with him. We may assume that $J_1(\alpha_{SR}) = \underline{x}\nu$ and $P_1(\alpha_{SR}) = 1$. Since his payoff is $(1-a_1)\underline{x}\nu$, he is better off when he assigns probability 1 to $\underline{a}$. By this analysis the following reduced problem serves the linear programming problem.

$$k^*(a, \boldsymbol{e}_1) := \max_{v_1, w_1(y)} v_1 \quad \text{subject to} \tag{2}$$

$$v_1 = (1-\delta)\underline{x}\nu f(1-\underline{a}) + \delta((1-p)w_1(0) + pw_1(\underline{a})), \tag{3}$$

$$v_1 \geq (1-\delta)\underline{x}\nu f(1-a_1) + \delta((1-p)w_1(0) + pw_1(a_1)) \quad \text{if } a_1 \neq \underline{a}, \tag{4}$$

$$v_1 \geq w_1(a_1) \qquad \text{for all } a_1 \in A_1. \tag{5}$$

Theorem 4.1 (Limit equilibrium payoff for only one capitalist) Suppose that $n = 1$. Then,

$$Q = \begin{cases} [0,\, \underline{x}\nu f(1-\frac{\underline{a}}{p})] & \text{if } p > \underline{a}, \\ \{0\} & \text{otherwise.} \end{cases} \qquad \square$$

Proof: When the capitalist 1 plays $\underline{a}$, the outcomes other than 0 or $\underline{a}$ never occur. We focus the continuation payoffs $w_1(0)$ and $w_1(\underline{a})$. The constraint (4) for $y_1 = 0$ implies that

$$v_1 \geq (1-\delta)\underline{x}\nu f + \delta w_1(0). \tag{6}$$

Since v_1 increases in $w_1(0)$ and $w_1(\underline{a})$ by (3), in order to maximize v_1 this constraint (6) must hold in equality. Let $\Pi_1 := \left(\begin{smallmatrix} 1 & 0 \\ 1-p & p \end{smallmatrix}\right)$ be the probability matrix which consists of two rows for $a = 0$ and $\underline{a}$ and two columns for $y = 0$ and $\underline{a}$. Since Π_1 is invertible by $p \neq 0$, we obtain $\Pi_1^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 1-1/p & 1/p \end{smallmatrix}\right)$.

$$\begin{aligned} \begin{pmatrix} w_1(0) \\ w_1(\underline{a}) \end{pmatrix} &= \frac{1}{\delta} v_1 \mathbf{1} - \frac{1-\delta}{\delta}\Pi_1^{-1}\begin{pmatrix} \underline{x}\nu f \\ \underline{x}\nu f(1-\underline{a}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\delta}v_1 - \frac{1-\delta}{\delta}\underline{x}\nu f \\ \frac{1}{\delta}v_1 - \frac{1-\delta}{\delta}\underline{x}\nu f(1-\underline{a}/p) \end{pmatrix}. \end{aligned} \tag{7}$$

Since $\underline{a}/p$ is positive, $w_1(0) < w_1(\underline{a})$. We obtain $v_1 = w_1(\underline{a})$ by (5). Then,

$$v_1 = w_1(\underline{a}_1) = \underline{x}\nu f\left(1 - \frac{\underline{a}}{p}\right), \tag{8}$$

$$w_1(0) = \underline{x}\nu f\left(1 - \frac{\underline{a}}{\delta p}\right). \tag{9}$$

The other continuation payoffs are arbitrary values between $0 \leq w_1(a_1) \leq v_1$. For instance, we can select $w_1(a_1) := w_1(0)$ for $a_1 \in A_1 \setminus \{0, \underline{a}\}$. If $p > \underline{a}$, by (8) the maximal score attainable by $a_1 = \underline{a}$ is nonnegative. Otherwise, the maximal score is 0 since the maximal score attainable by $a_1 = 0$ is clearly 0. ∎

Fudenberg and Levine did not consider the case $p \leq \underline{a}$. By Theorem 3.1, $\operatorname{int} Q \subset \lim_{\delta \to 1} E(\delta)$. Let us consider the boundary of Q. Since 0 is the stage game Nash equilibrium payoff, it is equilibrium payoff for any discount factor. Next corollary suggests that the upper boundary point $\underline{x}\nu f(1 - \underline{a}/p)$ occurs in the equilibrium for sufficiently large $\delta < 1$. The condition $a_1 \leq p$ is capitalist's incentive constraint for supplying positive dividend. The only action a_1 such that $1/f \leq a_1 \leq p$ satisfies both players' incentives for investment and payment. We can derive the value of δ such that every point of Q can be attainable in the equilibrium in the case of $p > \underline{a}$.

Corollary 4.1 If $p > \underline{a}$ in the one capitalist case,

$$[0, \underline{x}\nu f(1 - \underline{a}/p)] = E(\delta) \qquad \text{for } \delta \geq \frac{1}{2 - \underline{a}/p}. \qquad \square$$

Proof: Let W be $[0, \underline{x}\nu f(1 - \frac{\underline{a}}{p})]$. We seek the minimum δ under which W is self-decomposable by using $a_1^1 = 0$ and $\underline{a}$. Take an arbitrary vector $v_1 \in W$. When capitalist 1 plays $\underline{a}$, the continuation payoffs are the same as those in (7). In order to make each continuation payoff belong to W, it is sufficient that $v_1/\delta - (1-\delta)\underline{x}\nu f/\delta \geq 0$. When he plays $a_1^1 = 0$ and all investors consume directly, its continuation payoff is only one and equal to v_1/δ. If $v_1/\delta \leq \underline{x}\nu f(1 - \underline{a}/p)$, the continuation payoff v_1/δ lies in W. Since he can use $\underline{a}$ or 0 to decompose v_1 and v_1 belongs to W,

$$\max_{v_1 \in [0, \underline{x}\nu f(1-\underline{a}/p)]} \left\{ \min \left\{ \frac{\underline{x}\nu f - v_1}{\underline{x}\nu f}, \frac{v_1}{\underline{x}\nu f(1 - \underline{a}/p)} \right\} \right\} = \frac{1}{2 - \underline{a}/p}.$$

Since W is compact and convex, by locally self-decomposable lemma in Fudenberg, Levine, and Maskin (1994, p.1010) the conclusion holds. ∎

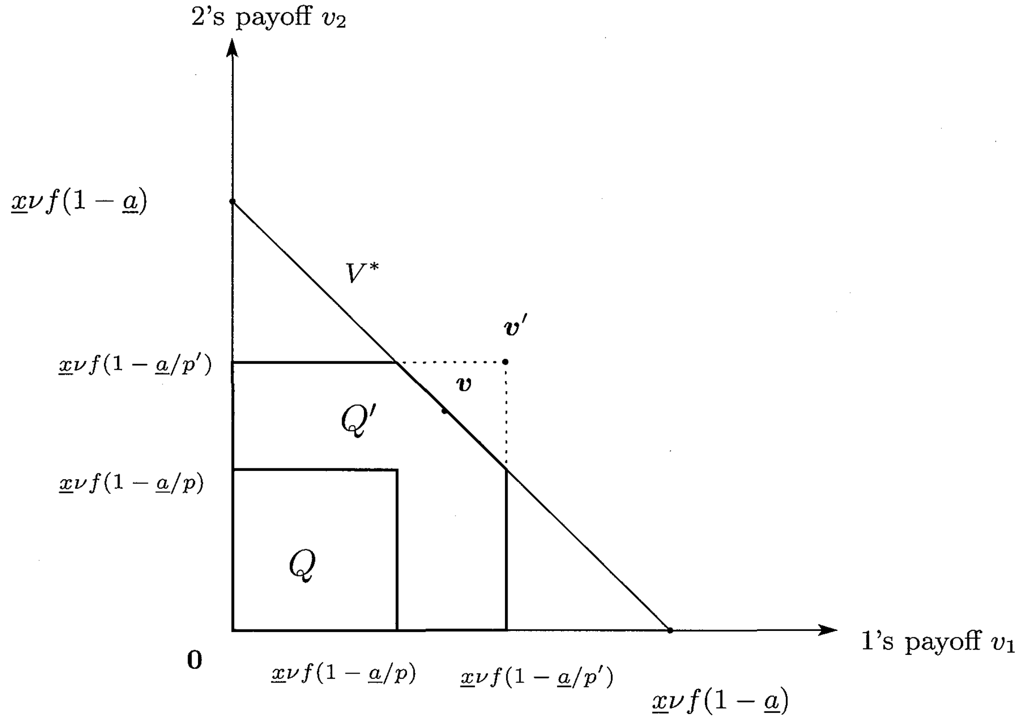

図 1　Two capitalists' Equilibrium Payoffs　($p < 2\underline{a}/(\underline{a}+1) \leq p'$)

Corollary 8.2 in Fudenberg and Levine (1994, p.131) argued that if $\delta \geq \underline{a}/p$, we can find an equilibrium yielding the maximal payoff $\underline{x}\nu f(1-\underline{a}/p)$. They only calculated δ such that $w_1(0)$ belongs to Q. However, in order to obtain an equilibrium including the maximal payoff we must find the condition under which Q or a compact convex set including it are self-decomposable. The Nash equilibrium is useful for decomposing payoffs near to 0.

Next, we analyze the maximal equilibrium payoffs for several long-run players. Given the number of players n and probability of success p, let $s^*(n;p)$ be the supremum of the sum of equilibrium payoffs when δ is sufficiently close to one.

Theorem 4.2 (Several capitalists case) Suppose that $n \geq 2$. If (i) $p \geq 2\underline{a}/(\underline{a}+1)$, then

$$s^*(n;p) = \underline{x}\nu f(1-\underline{a}) \qquad \text{for all } n \geq 2.$$

If (ii) $2\underline{a}/(\underline{a}+1) > p > \underline{a}$, then there exists $\overline{n} > 2$ such that

$$s^*(n;p) = n\underline{x}\nu f(1-\underline{a}/p) \qquad \text{for all } n < \overline{n},$$
$$s^*(n;p) = \underline{x}\nu f(1-\underline{a}) \qquad \text{for all } n \geq \overline{n}.$$

If (iii) $\underline{a} \geq p$, then

$$s^*(n;p) = 0 \qquad \text{for all } n \geq 2.$$

Proof: Let us consider the maximum half spaces in the positive coordinate directions. If the only one player i's payoff is maximized, by Theorem 4.1, his maximized payoff is $\underline{x}\nu f(1-\underline{a}/p)$. Since all maximal scores in all negative coordinate directions are 0 by Lemma 3.1, the intersection of all the maximum half spaces in the coordinate directions is

$$Q_c := \{\boldsymbol{v} \in \mathbb{R}^n_+ \mid v_i \leq \underline{x}\nu f(1-\underline{a}/p) \text{ for } i \in LR\}.$$

By Lemma 3.1, we see that $Q = V^* \cap Q_c$. Let the vector $\boldsymbol{v}'$ be the unique Pareto-efficient point in Q_c.

Let us consider the case (i) $p \geq 2\underline{a}/(\underline{a}+1)$. In this case for all $n \geq 2$,

$$\sum_{i \in LR} v_i' = n\underline{x}\nu f(1-\underline{a}/p) \geq \underline{x}\nu f(1-\underline{a}).$$

Since any Pareto-efficient vector $\boldsymbol{v} \in V^*$ satisfies $\sum_{i\in LR} v_i = \underline{x}\nu f(1-\underline{a})$ by Lemma 2.1, $\boldsymbol{v}'$ is not feasible in V^*. However, there exists a Pareto-efficient vector $\boldsymbol{v} \in V^*$ such that $\boldsymbol{v}' \geq \boldsymbol{v}$ and $\boldsymbol{v} \in Q_c$.[(2)] Therefore, by Lemma 3.1, a Pareto-efficient vector $\boldsymbol{v}$ also belongs to Q. Since any interior point $\boldsymbol{v}^*$ of Q is an equilibrium payoff by Theorem 3.1, for any $\varepsilon > 0$ there exists an equilibrium vector $\boldsymbol{v}^* \in \operatorname{int} Q$ such that $\sum_{i\in LR} v_i - \varepsilon = \underline{x}\nu f(1-\underline{a}) - \varepsilon < \sum_{i\in LR} v_i^*$. Then $\sup \sum_{i\in LR} v_i^* = \underline{x}\nu f(1-\underline{a})$.

Let us consider the case (ii) $2\underline{a}/(\underline{a}+1) > p > \underline{a}$. In this case we see that $2\underline{x}\nu f(1-\underline{a}/p) < \underline{x}\nu f(1-\underline{a})$ and there exists $\overline{n} > 2$ such that

$$n\underline{x}\nu f(1-\underline{a}/p) < \underline{x}\nu f(1-\underline{a}) \qquad \text{for all } n < \overline{n},$$
$$n\underline{x}\nu f(1-\underline{a}/p) \geq \underline{x}\nu f(1-\underline{a}) \qquad \text{for all } n \geq \overline{n}.$$

In first case $n < \overline{n}$, since $\boldsymbol{v}' \in \operatorname{int} V^*$, it is clear that $Q = Q_c$. Any Pareto-efficient vector cannot lie in Q. By Theorem 3.1 the supremum of the sum of equilibrium payoffs is $n\underline{x}\nu f(1-\underline{a}/p)$. The second case $n \geq \overline{n}$ is the same as the case (i).

In the case (iii) all the maximal scores in the positive coordinate directions are 0 as seen in the one capitalist case. ∎

By the assumption of $\underline{a}$ the critical point $2\underline{a}/(\underline{a}+1)$ is less than one. The equilibrium payoffs for two capitalists case is depicted in Figure 1. When p' is greater than the critical point, the capitalists' maximal joint profit $s^*(n;p')$ is unchangeable regardless of $n \geq 2$. A Pareto-efficient vector $\boldsymbol{v}$ is included by Q'. Any payoff near to the Pareto-efficient payoffs $\boldsymbol{v}$ can be decomposable by the actions such that all investment are concentrated on either capitalist 1 or 2. Therefore, nearly Pareto-efficient outcome can be equilibrium. The more the number of capitalists increases, the more the efficient point $\boldsymbol{v}'$ in Q increases. However, since the efficient point in Q is not feasible in V^*, $\boldsymbol{v}'$ cannot be equilibrium.

In the middle range of the probability p (case (ii)), when the number of capitalists is sufficiently small, the production and the dividend payment activities is not as profitable as these activities can achieve nearly Pareto-efficient payoffs. Especially, any nearly Pareto-efficient payoff cannot be decomposable in the positive coordinate direction. The more the number of capitalists increases, the more the efficient point $\boldsymbol{v}'$ in Q increases. The decomposability in the positive coordinate direction is more profitable. The implication induced by Fudenberg and Levine (1994, p.105)

> Rather, adding more capitalists promotes efficiency because the investors can 'punish' one capitalist by switching their lending to another capitalist, rather than withdrawing from the market.

holds in this case. Fudenberg and Levine (1994) defined the first best payoff by $\underline{x}\nu(f-1)$. This is greater than $\underline{x}\nu f(1-\underline{a})$ if $f\underline{a} > 1$. In this case the first best payoff is not feasible in V^*. They claimed that if there exists sufficient number of capitalists, then the first best payoff is attainable in their Corollary 8.1 (p.130). The Theorem 4.2 shows that if there exists sufficient number of capitalists, to the contrary of their claim, then a Pareto-efficient payoff $\underline{x}\nu f(1-\underline{a})$ is approximately attainable. Their claim only holds when $f\underline{a} = 1$. Their mistake results from taking no account of the set of the feasible, individually rational, and incentive compatible payoffs in the stage game. Since any equilibrium payoff lies in V^*, we must investigate the feasibility condition. By their method in Theorem 3.1 the Pareto-efficient outcome is attainable approximately.

5 Concluding Remarks

In the investment game when there exist sufficiently many capitalists and they are patient enough, the supremum of the sum of equilibrium payoffs for them attains a Pareto-efficient outcome with investor's incentive condition. In general it is different from the first best payoff. Whenever the probability of high output is sufficiently high, the supremum is unchangeable regardless of the number of capitalists if it is greater than one. The equilibrium characterization theorem in Fudenberg and Levine (1994) only supports nearly Pareto-efficient outcome.

References

[1] Fudenberg, D. and D. Levine (1994), "Efficiency and Observability with Long-run and Short-run Players." *Journal of Economic Theory* , **62**, 103–135.

[2] Fudenberg, D., D. Levine, and E. Maskin (1994), "The Folk Theorem with Imperfect Public Information." *Econometrica*, **62**, 997–1039.

[3] Fudenberg, D., and E. Maskin (1986), "Folk Theorem for Repeated Games with Discounting or with Incomplete Information," *Econometrica*, **54**, 533–554.